

Inhomogeneous charged pion condensation phenomenon in the NJL₂ model with quark number and isospin chemical potentials

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The properties of two-flavored massive Nambu–Jona-Lasinio model in (1+1)-dimensional space-time are investigated in the presence of isospin and quark number chemical potentials. The consideration is performed in the large- N_c limit, where N_c is the number of colored quarks. It is shown in the framework of this model that charged pion condensation phenomenon of dense quark/hadron isotopically asymmetric matter is rather a spatially inhomogeneous than a homogeneous one.

I. INTRODUCTION

Recently, much attention has been paid to the investigation of the QCD phase diagram in terms of baryonic as well as isotopic (isospin) chemical potentials. The reason is that dense baryonic matter which can appear in heavy-ion collision experiments has an evident isospin asymmetry. Moreover, the dense hadronic/quark matter inside compact stars is also expected to be isotopically asymmetric. To describe the above mentioned realistic situations, i.e. when the baryonic density is comparatively low, usually different nonperturbative methods or effective theories such as chiral effective Lagrangians and especially Nambu – Jona-Lasinio (NJL) type models [1] are employed. In this way, the QCD phase diagram including chiral symmetry restoration [2–6], color superconductivity [7–9], and charged pion condensation (PC) phenomena [10–14] were investigated under heavy-ion experimental and/or compact star conditions, i.e. in the presence of such external conditions as temperature, chemical potentials and possible external (chromo)magnetic fields (see the above references).

Obviously, the (3+1)-dimensional NJL models depend on the cutoff parameter which is typically chosen to be of the order of 1 GeV, so that the results of their usage are valid only at *comparatively low energies, temperatures and densities (chemical potentials)*. Moreover, there exists also a class of renormalizable theories, the (1+1)-dimensional chiral Gross–Neveu (GN) type models [15, 16], ¹ that can be used as a laboratory for the qualitative simulation of specific properties of QCD at *arbitrary energies*. Renormalizability, asymptotic freedom, as well as the spontaneous breaking of chiral symmetry (in vacuum) are the most fundamental inherent features both for QCD and all GN type models. In addition, the $\mu - T$ phase diagram is qualitatively the same for the QCD and GN models [17–20] (here μ is the quark number chemical potential and T is the temperature). Note also that the GN type models are suitable for the description of physics in quasi one-dimensional condensed matter systems like polyacetylene [21]. It is currently well understood (see, e.g., the discussion in [19, 20, 22]) that the usual *no-go* theorem [23], which generally forbids the spontaneous breaking of any continuous symmetry in two-dimensional spacetime does not work in the limit $N_c \rightarrow \infty$, where N_c is the number of colored quarks. This follows from the fact that in the limit of large N_c the quantum fluctuations, which would otherwise destroy a long-range order corresponding to a spontaneous symmetry breaking, are suppressed by $1/N_c$ factors. Thus, the effects inherent for real dense quark matter, such as Cooper pairing phenomenon (spontaneous breaking of the continuous $U(1)$ symmetry) or charged pion condensation (spontaneous breaking of the continuous isospin symmetry) might be simulated in terms of a simpler (1+1)-dimensional GN-type model, though only in the leading order of the large N_c approximation (see, e.g., [20, 24] and [25–27, 30], respectively).

This paper is devoted to investigation of the charged pion condensation (PC) phenomenon in the framework of the (1+1)-dimensional NJL model with two quark flavors and in the presence of the quark number (μ) as well as isospin (μ_I) chemical potentials. The consideration is performed in the leading order of the $1/N_c$ -expansion. In our previous papers [25–27] the phase diagram of the above mentioned massless or massive NJL₂ model was already investigated in the case of homogeneous, i.e. independent of space coordinate, order parameters (chiral and charged pion condensates). The situation corresponds to the conserved Lorentz and spatial translational invariance and is adequate to physical systems in vacuum, i.e. at zero chemical potentials. However, in dense baryonic matter, i.e. at nonzero quark number chemical potential, there might appear new phases with a spatially inhomogeneous chiral and/or charged pion condensates which destroy both chiral and/or isospin as well as spatial translational invariances of a system. In particular, the possibility of the phase with inhomogeneous chiral condensate was discussed in the framework of both (1+1)-dimensional [22, 28–30] and (3+1)-dimensional [31–38] models. At the same time the possibility of the phase with inhomogeneous charged pion condensation is less investigated. ²

Thus, in this paper, in contrast to [25–27], we consider the phase portrait of the above mentioned massive (1+1)-dimensional NJL model with two chemical potentials, μ and μ_I , in the leading order of the $1/N_c$ -expansion taking into

¹ Below we shall use the notation “NJL₂ model” instead of “chiral GN model” for (1+1)-dimensional models with a *continuous chiral symmetry*, since the chiral structure of the Lagrangian is the same as that of the (3+1)-dimensional NJL model.

² For example, spatially inhomogeneous ansatz for the charged pion condensate was investigated in the framework of the two-flavored NJL₄ model in [13]. It was claimed there that inhomogeneous PC phase is possible only at rather high values of an isotopic chemical potential, $\mu_I > \Lambda$, where $\Lambda \sim 0.65$ MeV is a cutoff parameter. So this result might be out of the scope of a model application. Moreover, the authors of [13] made some technical simplifications in their research of the inhomogeneous PC phenomenon.

account the possibility that the charged pion condensate might become spatially inhomogeneous. The temperature is taken to be zero. For simplicity, for the chiral condensate we use a spatially homogeneous ansatz. (In contrast, in our previous paper [30] the spatially inhomogeneous ansatz for the chiral condensate and homogeneous one for the charged pion condensate was used in the framework of the same massless NJL₂ model.) Notice once more, the isotopic asymmetry is an inevitable property of dense quark matter which might be created in heavy-ion collision experiments or inside compact stars. So, we believe that such a simplified study in the framework of the two-dimensional NJL model with isospin chemical potential could shed new light on the properties of real dense baryonic matter and will provide a deeper understanding of the charged PC phenomenon. In particular, it is shown in our paper that in the framework of the model under consideration a PC phase with *nonzero baryon density* is realized just in the case of inhomogeneous charged pion condensate but not in the case of homogeneous one. In analogy, one can expect that in real (3+1)-dimensional dense hadronic/quark matter the charged PC phenomenon is realized rather with inhomogeneous pion condensate than with spatially homogeneous one.

The paper is organized as follows. In Section II we derive, in the leading order of the large N_c -expansion, the general expression for the thermodynamic potential of the two-flavored massive NJL₂ model with quark number chemical potential μ and isospin chemical potential μ_I in the case of spatially homogeneous chiral condensate and inhomogeneous PC. First, in Sec. III we reduce our consideration to the case of homogeneous PC and find that in this case only a PC phase with *zero density* of quarks is possible in the model. Second, in Sec. IV it is shown that charged PC phase with *nonzero quark density* in the framework of the model is possible only with spatially inhomogeneous charged pion condensate. Final Sec. V presents a summary and some concluding remarks. The discussion of some technical problems are relegated to two Appendices.

II. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

We consider a (1+1)-dimensional NJL₂ model to mimic the phase structure of real dense quark matter composed of two massive quark flavors (u - and d - quarks). Its Lagrangian has the form:

$$L = \bar{q} \left[\gamma^\rho i \partial_\rho - m_0 + \mu \gamma^0 + \frac{\mu_I}{2} \tau_3 \gamma^0 \right] q + \frac{G}{N_c} \left[(\bar{q}q)^2 + (\bar{q}i\gamma^5 \vec{\tau} q)^2 \right], \quad (1)$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c -plet ($\alpha = 1, \dots, N_c$) as well as a two-component Dirac spinor (the summation in (1) over flavor, color, and spinor indices is implied); τ_k ($k = 1, 2, 3$) are Pauli matrices; the quark number chemical potential μ in (1) is responsible for the nonzero baryonic density of quark matter, whereas the isospin chemical potential μ_I is taken into account in order to study properties of quark matter at nonzero isospin densities (in this case the densities of u and d quarks are different). The Dirac gamma matrices in two-dimensional spacetime have the following form:

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \gamma^5 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Evidently, the model (1) is a simple generalization of the original (1+1)-dimensional Gross-Neveu model [15] with a single massless quark color N_c -plet to the case of two massive quark flavors and additional chemical potentials. As a result, in the case under consideration we have a modified flavor symmetry group, which depends essentially on whether the bare quark mass m_0 and isospin chemical potential μ_I take zero or nonzero values. Indeed, at $\mu_I = 0, m_0 = 0$ the Lagrangian (1) is invariant under transformations from the chiral $SU_L(2) \times SU_R(2)$ group. Then, at $\mu_I \neq 0, m_0 = 0$ this symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of the isospin operator (here and above the subscripts L, R mean that the corresponding group acts only on left, right handed spinors, respectively). Evidently, this symmetry can also be presented as $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is the isospin subgroup and $U_{AI_3}(1)$ is the axial isospin subgroup. Quarks are transformed under these subgroups as $q \rightarrow \exp(i\alpha\tau_3)q$ and $q \rightarrow \exp(i\alpha\gamma^5\tau_3)q$, respectively. In the case $m_0 \neq 0, \mu_I = 0$ the Lagrangian (1) is invariant with respect to the $SU_I(2)$, which is a diagonal subgroup of the chiral $SU_L(2) \times SU_R(2)$ group. Finally, in the most general case with $m_0 \neq 0, \mu_I \neq 0$ the initial model (1) is symmetric under the above mentioned isospin subgroup $U_{I_3}(1)$. In addition, in all foregoing cases the model is invariant under color $SU(N_c)$ -, baryon charge $U_B(1)$ - and electric charge $U_Q(1)$ groups.

The linearized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$), has the following form:

$$\tilde{L} = \bar{q} \left[\gamma^\rho i \partial_\rho - m_0 + \mu \gamma^0 + \frac{\mu_I}{2} \tau_3 \gamma^0 - \sigma - i\gamma^5 \pi_a \tau_a \right] q - \frac{N_c}{4G} \left[\sigma^2 + \pi_a \pi_a \right]. \quad (3)$$

From the Lagrangian (3) one obtains the following constraint equations for the bosonic fields

$$\sigma(x) = -2 \frac{G}{N_c} (\bar{q}q); \quad \pi_a(x) = -2 \frac{G}{N_c} (\bar{q}i\gamma^5 \tau_a q). \quad (4)$$

Obviously, the Lagrangian (3) is equivalent to the Lagrangian (1) when using the constraint equations (4). Furthermore, it is clear that the bosonic fields (4) are transformed under the isospin $U_{I_3}(1)$ subgroup in the following manner:

$$U_{I_3}(1): \quad \sigma \rightarrow \sigma; \quad \pi_3 \rightarrow \pi_3; \quad \pi_1 \rightarrow \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_2; \quad \pi_2 \rightarrow \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1, \quad (5)$$

i.e. the expression $(\pi_1^2 + \pi_2^2)$ remains unchanged under transformations of the isospin subgroup $U_{I_3}(1)$.

To avoid the *no-go* theorem [23], which forbids the spontaneous breaking of continuous symmetries in the considered case of one space dimension, we restrict the discussion only to the leading order of the large N_c expansion (i.e. to the case $N_c \rightarrow \infty$), where this theorem is not valid [19, 20, 22]. In particular, the effective action $S_{\text{eff}}(\sigma, \pi_a)$ can be found in this approximation through the relation:

$$\exp(iS_{\text{eff}}(\sigma, \pi_a)) = N' \int [d\bar{q}][dq] \exp\left(i \int \tilde{L} d^2x\right), \quad (6)$$

where N' is a normalization constant. It is clear from (3) and (6) that

$$S_{\text{eff}}(\sigma, \pi_a) = -N_c \int \frac{\sigma^2 + \pi_a^2}{4G} d^2x + \tilde{S}_{\text{eff}}, \quad (7)$$

where the quark contribution to the effective action, i.e. the term \tilde{S}_{eff} in (7), is given as follows

$$\exp(i\tilde{S}_{\text{eff}}) = N' \int [d\bar{q}][dq] \exp\left(i \int \bar{q}[\gamma^\rho \partial_\rho - m_0 + \mu\gamma^0 + \nu\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a]q d^2x\right). \quad (8)$$

Here we used the notation $\nu = \mu_I/2$. The ground state expectation values $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ of the composite bosonic fields are determined by the saddle point equations,

$$\frac{\delta S_{\text{eff}}}{\delta\sigma(x)} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta\pi_a(x)} = 0, \quad (9)$$

where $a = 1, 2, 3$. In vacuum, i.e. in the state corresponding to an empty space with zero particle density and zero values of the chemical potentials μ and μ_I , the quantities $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ do not depend on space coordinates. However, in dense quark medium, when $\mu \neq 0$, $\mu_I \neq 0$, the ground state expectation values of bosonic fields might have a nontrivial dependence on x . In particular, in this paper we will use the following ansatz:

$$\langle\sigma(x)\rangle = M - m_0, \quad \langle\pi_3(x)\rangle = 0, \quad \langle\pi_1(x)\rangle = \Delta \cos(2bx), \quad \langle\pi_2(x)\rangle = \Delta \sin(2bx), \quad (10)$$

where M, b , and Δ are constant quantities. In fact, they are coordinates of the global minimum point of the thermodynamic potential (TDP) $\Omega(M, b, \Delta)$.³ In the leading order of the large N_c -expansion it is defined by the following expression:

$$\int d^2x \Omega(M, b, \Delta) = -\frac{1}{N_c} S_{\text{eff}}(\sigma(x), \pi_a(x)) \Big|_{\sigma(x)=\langle\sigma(x)\rangle, \pi_a(x)=\langle\pi_a(x)\rangle}, \quad (11)$$

which gives

$$\int d^2x \Omega(M, b, \Delta) = \int d^2x \frac{(M - m_0)^2 + \Delta^2}{4G} + \frac{i}{N_c} \ln \left(\int [d\bar{q}][dq] \exp\left(i \int d^2x \bar{q} \mathcal{D} q\right) \right), \quad (12)$$

where

$$\bar{q} \mathcal{D} q = \bar{q}(\gamma^\rho i \partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - M)q - \Delta(\bar{q}_u i\gamma^5 q_d) e^{-2ibx} - \Delta(\bar{q}_d i\gamma^5 q_u) e^{2ibx}. \quad (13)$$

(Remember, in this formula q is indeed a flavor doublet, i.e. $q = (q_u, q_d)^T$.) To proceed, let us introduce in (12)-(13) the new quark doublets, ψ and $\bar{\psi}$, namely: $\psi = \exp(i\tau_3 bx)q$ and $\bar{\psi} = \bar{q} \exp(-i\tau_3 bx)$. Since this transformation of quark fields does not change the path integral measure in (12), the expression (12) for the thermodynamic potential is easily transformed to the following one:

$$\int d^2x \Omega(M, b, \Delta) = \int d^2x \frac{(M - m_0)^2 + \Delta^2}{4G} + \frac{i}{N_c} \ln \left(\int [d\bar{\psi}][d\psi] \exp\left(i \int d^2x \bar{\psi} D \psi\right) \right), \quad (14)$$

³ Here and in what follows we will use a rather conventional notation "global" minimum in the sense that among all our numerically found local minima the thermodynamical potential takes in their case the lowest value. This does not exclude the possibility that there exist other inhomogeneous condensates, different from (10), which lead to ground states with even lower values of the TDP.

where instead of the x -dependent Dirac operator \mathcal{D} a new x -independent operator D appears

$$D = \gamma^\nu i\partial_\nu - M + \mu\gamma^0 + \tau_3\gamma^1 b + \nu\tau_3\gamma^0 - i\Delta\tau_1\gamma^5. \quad (15)$$

The expression (14) for the thermodynamic potential is easily transformed to the following one:

$$\begin{aligned} \Omega(M, b, \Delta) &= \frac{(M - m_0)^2 + \Delta^2}{4G} + i \frac{\text{Tr}_{sf x} \ln D}{N_c \int d^2 x} \\ &= \frac{(M - m_0)^2 + \Delta^2}{4G} + i \text{Tr}_{sf} \int \frac{d^2 p}{(2\pi)^2} \ln \overline{D}(p), \end{aligned} \quad (16)$$

where the $\text{Tr}_{sf x}$ stands for the trace in spinor- (s), flavor- (f) as well as two-dimensional coordinate- (x) spaces, respectively, and Tr_{sf} is the respective trace without x -space. Moreover, $\overline{D}(p) = \not{p} + \mu\gamma^0 + \tau_3\gamma^1 b + \nu\tau_3\gamma^0 - M - i\gamma^5\Delta\tau_1$ is the momentum space representation of the Dirac operator D (15). Obviously, $\overline{D}(p)$ is a 4×4 matrix in the direct product of the spinor and flavor spaces. Since $\text{Tr}_{sf} \ln \overline{D}(p) = \ln \det \overline{D}(p)$, one can evaluate the expression (16) with a help of any program of analytical calculations and find

$$\Omega(M, b, \Delta) \equiv \Omega^{un}(M, b, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} + i \int \frac{d^2 p}{(2\pi)^2} \ln \det \overline{D}(p), \quad (17)$$

where

$$\begin{aligned} \det \overline{D}(p) &= \Delta^4 + 2\Delta^2(M^2 + p_1^2 + \nu^2 - b^2 - \eta^2) \\ &\quad + (M^2 + (p_1 - b)^2 - (\eta + \nu)^2)(M^2 + (p_1 + b)^2 - (\eta - \nu)^2) \end{aligned} \quad (18)$$

and $\eta = p_0 + \mu$. (In order to emphasize the fact that the expression (17) is divergent, i.e. unrenormalized, we use in this TDP notation the superscript “un“.) Obviously, the function $\Omega^{un}(M, b, \Delta)$ is symmetric with respect to the transformation $\Delta \rightarrow -\Delta$. (At $m_0 = 0$ it is also invariant with respect to the $M \rightarrow -M$ transformation.) Moreover, it is invariant under each of the transformations $b \rightarrow -b$, $\mu \rightarrow -\mu$ and $\nu \rightarrow -\nu$.⁴ Hence, without loss of generality, we restrict ourselves to the constraints $\Delta \geq 0$, $\mu \geq 0$, $b \geq 0$, and $\nu \geq 0$. In the following, we will investigate the global minimum point of the TDP (17) just on this region. However, first of all let us consider the case of spatially homogeneous condensates, i.e. the $b = 0$ case.

III. THE CASE OF HOMOGENEOUS CHARGED PION CONDENSATE, $b = 0$

Supposing that $b = 0$ in (17), we obtain after some technical calculations the TDP for the case of spatially homogeneous charged pion condensate,

$$\Omega^{un}(M, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} + i \int \frac{d^2 p}{(2\pi)^2} \ln \left\{ \left[(p_0 + \mu)^2 - (E_\Delta^+)^2 \right] \left[(p_0 + \mu)^2 - (E_\Delta^-)^2 \right] \right\}, \quad (19)$$

where

$$E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2}, \quad E^\pm = E \pm \nu, \quad \nu = \mu_I/2, \quad E = \sqrt{p_1^2 + M^2}. \quad (20)$$

The argument of the $\ln(x)$ -function in (19) is proportional to the inverse quark propagator in the energy-momentum space representation. Hence, its zeros are the poles of the quark propagator. So, using (19) one can find the dispersion laws for quasiparticles, i.e. the momentum dependence of the quark (p_{0u} , p_{0d}) and antiquark ($p_{0\bar{u}}$, $p_{0\bar{d}}$) energies, in a medium (the full expression of the quark propagator matrix is presented in Appendix B of paper [26]):

$$p_{0u} = E_\Delta^- - \mu, \quad p_{0d} = E_\Delta^+ - \mu, \quad p_{0\bar{u}} = -(E_\Delta^+ + \mu), \quad p_{0\bar{d}} = -(E_\Delta^- + \mu). \quad (21)$$

Integrating in (19) over p_0 (see in [25] for similar integrals), one obtains for the unrenormalized TDP of the system at zero temperature the following expression:

$$\begin{aligned} \Omega^{un}(M, \Delta) &= \frac{(M - m_0)^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left\{ E_\Delta^+ + E_\Delta^- \right. \\ &\quad \left. + (\mu - E_\Delta^+) \theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-) \theta(\mu - E_\Delta^-) \right\}, \end{aligned} \quad (22)$$

⁴ Indeed, if simultaneously with $b \rightarrow -b$ or $\mu \rightarrow -\mu$ transformations we perform in the integral (17) the $p_1 \rightarrow -p_1$ or $p_0 \rightarrow -p_0$ change of variables, respectively, then one can easily see that the expression (17) remains intact. Finally, if $\nu \rightarrow -\nu$, we should transform $p_1 \rightarrow -p_1$ in the integral (17) in order to be convinced that the TDP remains unchanged.

where $\theta(x)$ is the Heaviside theta-function. It is clear that the TDP (22) is an ultraviolet divergent quantity, so in order to get any physical information one should renormalize it, using a special dependence of such quantities as the bare coupling constant G and the bare quark mass m_0 on the cutoff parameter Λ (Λ restricts the integration region in the divergent integral in (22), $|p_1| < \Lambda$). The renormalization procedure for the simplest massive GN model was already discussed in the literature, see, e.g., in [18, 19, 26]. In a similar way, it is easy to see that, cutting of the divergent integral in (22) and then using the substitution $G \equiv G(\Lambda)$ and $m_0 \equiv mG(\Lambda)$, where

$$\frac{1}{2G(\Lambda)} = \frac{2}{\pi} \ln \left(\frac{2\Lambda}{M_0} \right) \quad (23)$$

and m, M_0 are new free finite renormalization group invariant massive parameters⁵ (which do not depend on the cutoff Λ), it is possible to obtain in the limit $\Lambda \rightarrow \infty$ a finite renormalization group invariant expression for the TDP (for details see, e.g., the papers [26]). Namely,

$$\begin{aligned} \Omega^{ren}(M, \Delta) = & V_0(M, \Delta) - \frac{mM}{2} - \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \left\{ E_{\Delta}^{+} + E_{\Delta}^{-} - 2\sqrt{p_1^2 + M^2 + \Delta^2} \right. \\ & \left. + (\mu - E_{\Delta}^{+})\theta(\mu - E_{\Delta}^{+}) + (\mu - E_{\Delta}^{-})\theta(\mu - E_{\Delta}^{-}) \right\}, \end{aligned} \quad (24)$$

where

$$V_0(M, \Delta) = \frac{M^2 + \Delta^2}{2\pi} \left[\ln \left(\frac{M^2 + \Delta^2}{M_0^2} \right) - 1 \right] \quad (25)$$

is a finite renormalization group invariant expression for the TDP (24) in vacuum, i.e. at $\mu = 0$ and $\mu_I = 0$, taken in the chiral limit, i.e. at $m = 0$.

In the following, when studying the phase structure, the quantity M_0 is still treated as a free parameter, however instead of the massive parameter m of the model we will use a dimensionless parameter, $\tilde{\alpha} \equiv \pi m/M_0$. As a result, one can see that in the massive NJL₂ model the dimensional transmutation effect is absent formally. Indeed, both before and after renormalization this massive model is parameterized by one massive- and one dimensionless quantity. (Before renormalization the model is characterized by a bare mass m_0 and a dimensionless bare coupling constant G , while after renormalization the mass M_0 and dimensionless quantity $\tilde{\alpha}$ are free model parameters.) In contrast, in the massless GN-type models, i.e. at $m_0 = 0$, the coupling constant G is replaced after renormalization by the massive parameter M_0 (it is the so-called dimensional transmutation phenomenon).

In our subsequent calculations throughout the paper the quantity $\tilde{\alpha}$ is fixed by $\tilde{\alpha} = \tilde{\alpha}_0 \approx 0.17$. In this case we have in the initial NJL₂ model the same relation between the pion mass and the dynamical quark mass in vacuum as in some NJL-type models in the realistic case of the (3+1)-spacetime [26].

Investigating the behavior of the global minimum point (whose coordinates are just the gaps M and Δ) of the TDP (24) vs chemical potentials, it is possible to establish the phase structure presented in Fig. 1. There, in the phases 1, 2 and 3 the gap Δ is vanishing, i.e. these are the normal quark matter phases with a nonzero gap M . However, at the boundaries between phases the gap M changes its value by a jump (the details of the investigation, including the behavior of gaps, particle densities, meson masses etc, are presented in [26]). For each point (ν, μ) of the vacuum region of Fig. 1 we have $\Delta = 0$ and $M \approx 1.04M_0$ (the physical meaning of the parameter M_0 is described in footnote 5). Finally, one can see in Fig. 1 the homogeneous charged pion condensation phase in which both gaps are not equal to zero. What is more interesting for us is that all over this phase the quark number density $n_q = -\partial\Omega^{ren}(M, \Delta)/\partial\mu$ is equal to zero, $n_q = 0$.

Therefore, in dense (i.e. with nonzero n_q) quark matter, mimicked by the initial GN-type model, the phase with spatially homogeneous charged pion condensation can not be realized.

IV. INHOMOGENEOUS ANSATZ FOR THE CHARGED PION CONDENSATE, $b \neq 0$

In this Section, the possibility for the ground state of the initial NJL₂ model (1) with spatially inhomogeneous charged pion condensate is investigated. We start with the unrenormalized TDP (17) which can be rewritten in the form

$$\Omega^{un}(M, b, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} + i \int \frac{d^2p}{(2\pi)^2} \ln(\eta^4 + A\eta^2 + B\eta + C), \quad (26)$$

⁵ Note, the quantity m is not equal to the physical, or dynamical, quark mass M . The last one is defined by the pole position of the quark propagator. Alternatively, it can be found as a gap, i.e. one of the coordinates of the global minimum point of the thermodynamic potential. However, parameter M_0 is equal to a dynamically generated quark mass in the vacuum and at $m_0 = 0$ (a more detailed discussion on the physical essence of these parameters is given in [26]).

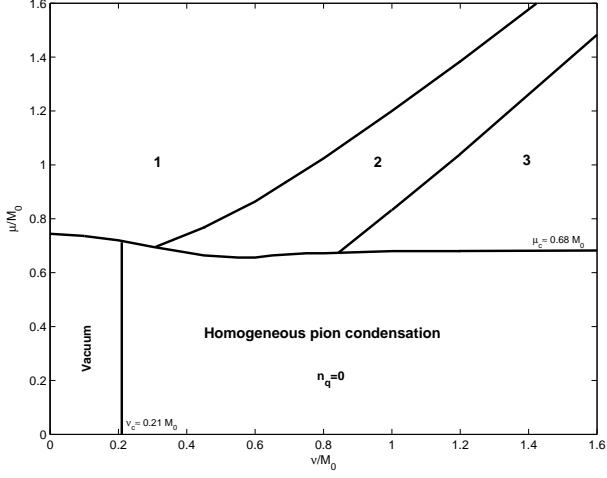


FIG. 1. The (μ, ν) phase portrait of the model considered at $T = 0$ and $\nu \equiv \mu_I/2 > 0$ in the case of spatially homogeneous condensates. Numbers 1, 2 and 3 denote different normal quark matter phases with $\Delta = 0$, $M \neq 0$. On all the lines of the Figure, first order phase transitions occur except for the boundary between vacuum and homogeneous pion condensation phase, where a second order phase transition takes place. n_q is the quark number density.

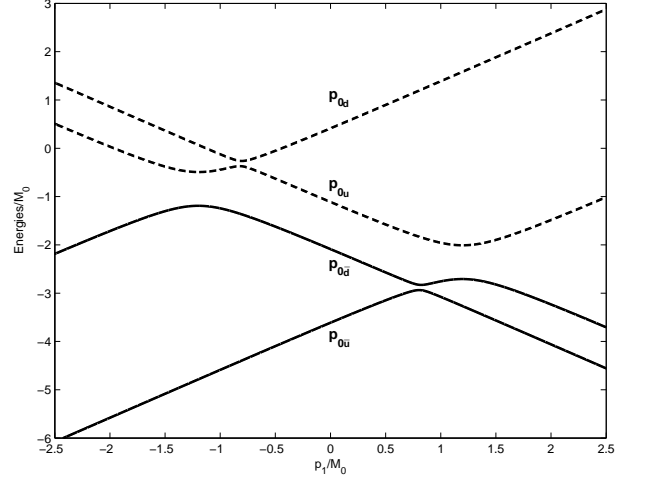


FIG. 2. The quasiparticle energies (29) vs p_1 at $\mu = 0.8M_0$, $\nu = 1.2M_0$, $\Delta = 0.35M_0$, $M = 0.06M_0$, and $b = 0.76M_0$.

where the notation $\eta = p_0 + \mu$ as well as the identity $\det \overline{D}(p) \equiv \eta^4 + A\eta^2 + B\eta + C$ with

$$\begin{aligned} A &= -2(M^2 + b^2 + p_1^2 + \nu^2 + \Delta^2), \quad B = -8p_1b\nu, \\ C &= (M^2 + b^2 + p_1^2 + \nu^2 + \Delta^2)^2 - 4(p_1^2\nu^2 + b^2\nu^2 + \Delta^2b^2 + M^2\nu^2 + p_1^2b^2) \end{aligned} \quad (27)$$

are used. The argument of the \ln -function in (26) can be expanded into a product of four linear multipliers,

$$\Omega^{un}(M, b, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} + i \int \frac{d^2p}{(2\pi)^2} \ln [(\eta - \eta_1^+)(\eta - \eta_1^-)(\eta - \eta_2^+)(\eta - \eta_2^-)], \quad (28)$$

where η_k^\pm are presented in Appendix A by the expressions (A1). If the quantities A, B and C in (26) are defined by the relations (27), then numerical analysis shows that all the roots η_k^\pm are real valued quantities vs gaps M, Δ, b , chemical potentials μ, μ_I and spatial momentum p_1 . Moreover, two of the roots, η_2^\pm , are negative valued quantities. Taking into account the remark after formula (20), it is possible to obtain immediately from (28) the quark-antiquark dispersion laws,

$$p_{0u} = \eta_1^- - \mu, \quad p_{0d} = \eta_1^+ - \mu, \quad p_{0\bar{u}} = \eta_2^- - \mu, \quad p_{0\bar{d}} = \eta_2^+ - \mu. \quad (29)$$

Note that at $b = 0$ the quasiparticle energies (29) coincide with the corresponding expressions from (21). At the particular values of the chemical potentials and gaps the plots of the quasiparticle energies p_{0u}, \dots (29) vs p_1 are presented in Fig. 2.

Now, it is possible to perform the p_0 -integration in (28) using the general formula (see Appendix B)

$$\int_{-\infty}^{\infty} dp_0 \ln(p_0 - a) = i\pi|a|, \quad (30)$$

where a is a real quantity. As a result, we have

$$\Omega^{un}(M, b, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} [|\mu - \eta_1^+| + |\mu - \eta_1^-| + |\mu - \eta_2^+| + |\mu - \eta_2^-|]. \quad (31)$$

To renormalize the expression (31) we must first regularize it. In this connection, it is necessary to make the following remark. In the case of homogeneous condensates (see the previous Section) usually the momentum cutoff regularization scheme is used. However it does not work in the case of spatially inhomogeneous condensates since three-momentum is no longer conserved⁶. As discussed in the recent papers [30, 32, 33, 35], an adequate regularization scheme in the

⁶ If the momentum cutoff regularization is used in the inhomogeneous case, the TDP acquires some non-physical properties such as unboundedness from below with respect to b , etc. As a result, an additional modification of the TDP is needed (for details see in [13, 30, 38]).

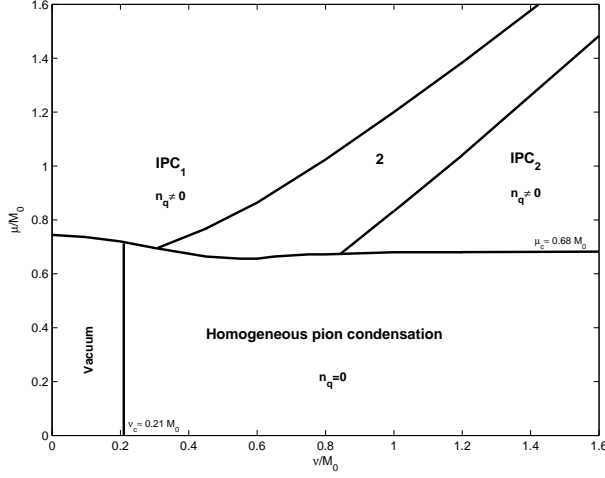


FIG. 3. Phase portrait of the model when inhomogeneity of pion condensates is taken into account. Here $IPC_{1,2}$ denote inhomogeneous charged pion condensation phases, n_q is a quark number density, 2 is a normal quark matter phase with $\Delta = 0$, $M \neq 0$ (see in Fig. 1).

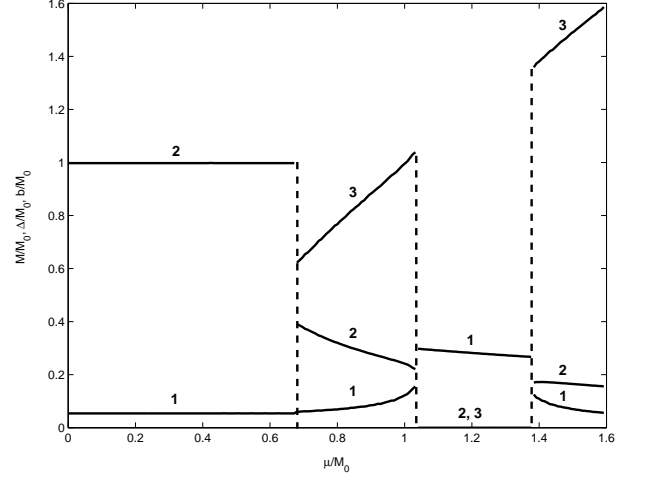


FIG. 4. Gaps M (line 1), Δ (line 2) and inhomogeneity wave vector b (line 3) vs μ at fixed $\nu = 1.2M_0$.

case of spatially inhomogeneous phases is that with an energy constraint equal for all quasiparticles. So, dealing with spatial inhomogeneity, one can use, e.g., the Schwinger proper-time regularization, dimensional regularization etc. In particular, in our recent paper [35] the *symmetric energy cutoff regularization* scheme was proposed in considering the behavior of chiral density waves in the presence of an external magnetic field in the framework of the four-dimensional Nambu–Jona-Lasinio model. There, for each quasiparticle the same (finite) interval of their energy values was allowed to contribute to the regularized thermodynamic potential. In the present investigation we will also use the symmetric energy cutoff regularization for the TDP, i.e.

$$\Omega^{un}(M, b, \Delta) \longrightarrow \Omega^{reg}(M, b, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} [|p_{0u}| \theta(\Lambda - |p_{0u}|) + |p_{0d}| \theta(\Lambda - |p_{0d}|) + |p_{0\bar{u}}| \theta(\Lambda - |p_{0\bar{u}}|) + |p_{0\bar{d}}| \theta(\Lambda - |p_{0\bar{d}}|)], \quad (32)$$

where the notations (29) for quasiparticle energies p_{0u} etc. are used. Now, let us consider the identity

$$\Omega^{reg}(M, b, \Delta) = \left(\Omega^{reg}(M, b, \Delta) - \Omega^{reg}(M, b, \Delta)|_{b=0, \mu=0, \nu=0} \right) + \Omega^{reg}(M, b, \Delta)|_{b=0, \mu=0, \nu=0}. \quad (33)$$

Clearly, at $b = 0, \mu = 0, \nu = 0$ for all quantities $\eta_{1,2}^{\pm}$ one finds the relation $|\eta_{1,2}^{\pm}| = \sqrt{p_1^2 + M^2 + \Delta^2}$, so that

$$\Omega^{reg}(M, b, \Delta)|_{b=0, \mu=0, \nu=0} = \frac{(M - m_0)^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{\pi} \sqrt{p_1^2 + M^2 + \Delta^2} \theta \left(\Lambda - \sqrt{p_1^2 + M^2 + \Delta^2} \right). \quad (34)$$

Since the expression in parenthesis in (33) is an ultraviolet (UV) convergent one, i.e. it is a finite quantity in the $\Lambda \rightarrow \infty$ limit, we see that in (33) all the UV divergences are located in the last term which is nothing but energy cutoff regularized vacuum thermodynamic potential of the system (34). Hence, in order to renormalize the TDP $\Omega^{un}(M, b, \Delta)$ (31) it is sufficient to remove UV divergences from the quantity (34) by substituting in (34) $G \equiv G(\Lambda)$ and $m_0 = mG(\Lambda)$ by quantities with an appropriate behavior of $G(\Lambda)$ vs Λ . As a result, we have

$$\Omega^{un}(M, b, \Delta) \longrightarrow \Omega^{ren}(M, b, \Delta) = V_0(M, \Delta) - \frac{mM}{2} - \lim_{\Lambda \rightarrow \infty} \left\{ \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} \left[|p_{0u}| \theta(\Lambda - |p_{0u}|) + |p_{0d}| \theta(\Lambda - |p_{0d}|) + |p_{0\bar{u}}| \theta(\Lambda - |p_{0\bar{u}}|) + |p_{0\bar{d}}| \theta(\Lambda - |p_{0\bar{d}}|) - 4\sqrt{p_1^2 + M^2 + \Delta^2} \theta \left(\Lambda - \sqrt{p_1^2 + M^2 + \Delta^2} \right) \right] \right\}. \quad (35)$$

where $V_0(M, \Delta)$ is given in (25).

We have studied numerically the TDP (35) as a function of M , Δ and b for some physically motivated value of the massive parameter $m = M_0 \tilde{\alpha}_0 / \pi$, where $\tilde{\alpha}_0 \approx 0.17$ and M_0 is a free parameter of the model (see the corresponding explanation at the end of the previous section III). The properties of its global minimum point vs chemical potentials give us the phase structure of the model which is presented in Fig. 3. It is easy to see that the phase structure in the case of spatially inhomogeneous condensates (see Fig. 3) is the same as in the case of homogeneous ones (see Fig.

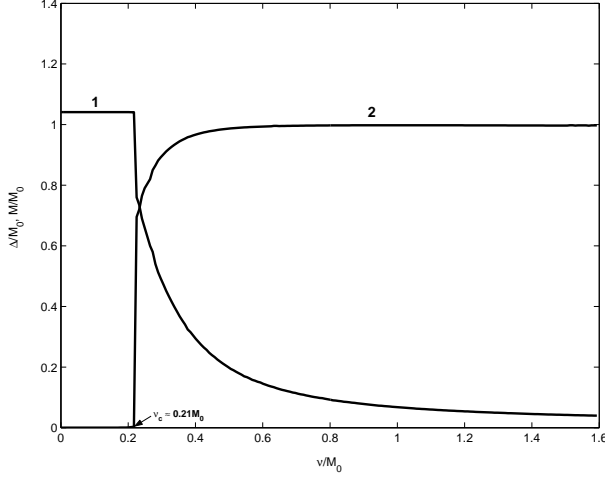


FIG. 5. Gaps M (line 1) and Δ (line 2) vs ν at fixed $\mu = 0.3M_0$. In this case $b \equiv 0$.

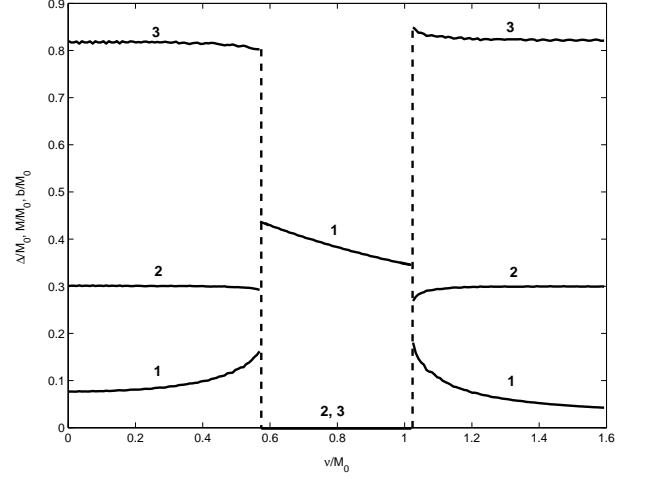


FIG. 6. Gaps M (line 1), Δ (line 2) and inhomogeneity wave vector b (line 3) vs ν at fixed $\mu = 0.85M_0$.

1) but with two essential exceptions. Namely, the normal quark matter phases 1 and 3 of Fig. 1 are replaced by two inhomogeneous pion condensation phases $\text{IPC}_{1,2}$ in Fig. 3. The behavior of the gaps M and Δ as well as of the wave vector b vs chemical potentials in these phases are presented in Figs 4-6. The TDP (35) provides also the expressions for the quark number density n_q and isospin density n_I ,

$$n_q = -\frac{\partial \Omega^{\text{ren}}(M, b, \Delta)}{\partial \mu}, \quad n_I = -\frac{\partial \Omega^{\text{ren}}(M, b, \Delta)}{\partial \mu_I}. \quad (36)$$

In particular, as it follows from (36) and (35)

$$\begin{aligned} n_q &= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} [\theta(\mu - \eta_1^+) + \theta(\mu - \eta_1^-) + \theta(\mu - \eta_2^+) + \theta(\mu - \eta_2^-) - 2] \\ &= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} [\theta(\mu - \eta_1^+) + \theta(\mu - \eta_1^-)], \end{aligned} \quad (37)$$

where the last equality appears due to the fact that $\eta_2^+ < 0$. Using these expressions one can easily prove that in the inhomogeneous pion condensation phases $\text{IPC}_{1,2}$ both isospin density n_I and quark number density n_q are nonzero (see Fig. 3). In contrast, in the homogeneous pion condensation phase of Fig. 3 the density n_q is zero. The behavior of n_I and n_q at some particular values of chemical potentials are presented in Figs 7, 8. Hence, we have proved that in the framework of the initial model the charged PC phenomenon with nonzero n_q -density is possible only with spatially inhomogeneous charged pion condensate.

V. SUMMARY AND CONCLUSIONS

This paper is devoted to the investigation of the so-called charged pion condensation (PC) phenomenon which might be observed in dense baryonic matter with different contents of u and d quarks. To simplify the consideration, we have restricted ourselves to the (1+1)-dimensional NJL-type model (1) with quark number μ and isospin μ_I chemical potentials at zero temperature. Special attention is paid to the influence of spatial inhomogeneity of different condensates on charged PC phenomenon. Our consideration is performed in the leading order of the large- N_c expansion.

Recall, the charged PC phenomenon was studied recently in the framework of some QCD-like effective theories such as NJL models or chiral effective theories in the usual (3+1)-dimensional Minkowski spacetime [10–14]. However, the existence of the charged PC phase with *nonzero baryon or quark number density*, denoted below as PCd phase, was there predicted without sufficient certainty. Indeed, for some values of model parameters (the coupling constant G , cutoff parameter Λ etc.) the PCd phase is allowed by NJL models. However, for other physically interesting values of G and Λ the PCd phase is forbidden in the framework of NJL models [11]. Moreover, if the electric charge neutrality constraint is imposed, the charged pion condensation phenomenon depends strongly on the bare (current) quark mass values. In particular, it turns out that the PCd phase is forbidden in the framework of NJL models if bare quark masses reach the physically acceptable values of $5 \div 10$ MeV (see in [14]).

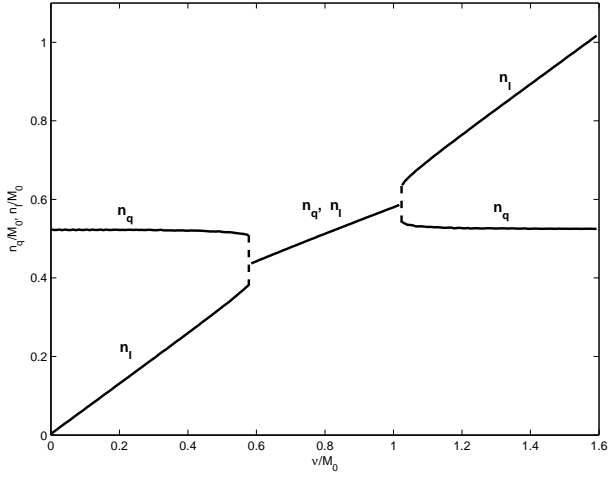


FIG. 7. Quark number, n_q , and isospin, n_I , densities vs ν at $\mu = 0.85M_0$.

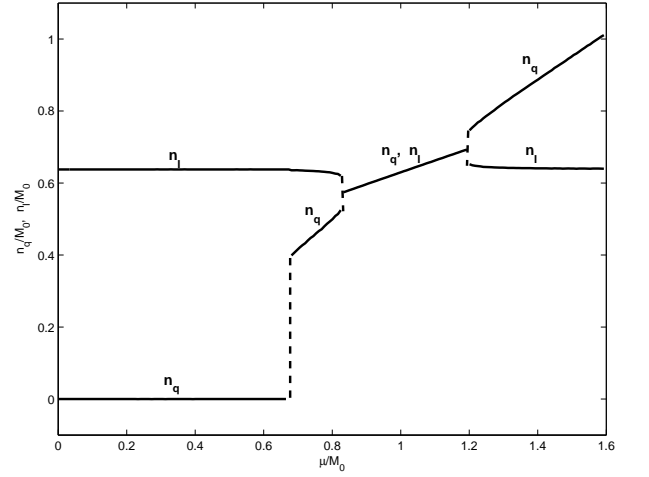


FIG. 8. Quark number, n_q , and isospin, n_I , densities vs μ at $\nu = M_0$.

As for investigations of the charged pion condensation phenomenon in the framework of the (1+1)-dimensional massive/massless NJL model (1), the results of our recent papers show that the PCd phase is also absent there, if PC condensate is spatially homogeneous [26, 30] (see also Sec. III of the present paper). However, earlier we have found one factor which promotes the creation of the PCd phase at least in (1+1)-dimensional spacetime. It is the finiteness of the volume of a physical system [39]. Since such a constraint with certain boundary conditions imposed for any system is equivalent to its consideration in a space with nontrivial topology, we have studied in [39] the initial model (1) in the spacetime $R^1 \times S^1$ (spatial coordinate is compactified) and proved that at some boundary conditions for spinor fields the charged PCd phase is realized in the system.

In this paper, we have proved that a spatial inhomogeneity of PC condensate is also a factor which promotes the appearance of PCd phases on the phase diagram of the NJL₂ model (1). Indeed, if consideration of dense quark matter is performed in terms of homogeneous pion condensates, then PCd phase is absent on the phase diagram of the model (1) (see Fig. 1). However, if the spatial modulation of pion condensates is taken into account in the form (10), then two PCd phases appear on the phase diagram of the model (1) (those are IPC₁ and IPC₂ phases of Fig. 3).

In summary, we conclude that charged pion condensation phenomenon of dense and isotopically asymmetric quark/hadron matter is more preferable to be spatially inhomogeneous than homogeneous.

Finally, we would like to discuss the reliability of the main result of our paper and try to predict what might happen with the PCd phase in the framework of a more general ansatz for condensates. To simplify the problem, let us consider the case of massless NJL₂ model (1) with $m_0 = 0$, with an evident generalization of the ansatz (10). Indeed, if $m_0 = 0$, then it is possible to study the phase structure of the model in terms of the following simultaneous spatial modulations of the chiral and pion condensates

$$\langle \sigma(x) \rangle = M \cos(2ax), \quad \langle \pi_3(x) \rangle = M \sin(2ax), \quad \langle \pi_1(x) \rangle = \Delta \cos(2bx), \quad \langle \pi_2(x) \rangle = \Delta \sin(2bx). \quad (38)$$

Evidently, there are three particular cases of (38). i) The choice $a = b = 0$ corresponds to spatially homogeneous condensates and the phase structure of the model for this parameterization was studied in [26]. ii) Then, the choice $a = 0$ is really the ansatz (10) at $m_0 = 0$. iii) Finally, the phase structure of the model under the constraint $b = 0$ was studied in [30]. Recall that in cases i) and iii) the PCd phase does not appear at the phase diagram. We have made preliminary estimations of the phase structure of the massless model (1) in the framework of the ansatz (38) and found that 1) the phase with $M \neq 0$, $a \neq 0$, $\Delta \neq 0$, and $b \neq 0$ is absent. 2) There is an absolute minimum of the TDP corresponding to spatially inhomogeneous PCd phase with $\Delta \neq 0$, $b \neq 0$, $M = 0$, $a = 0$. 3) For the same values of chemical potentials there is an equivalent TDP extremum, corresponding to a chiral spiral phase, where $M \neq 0$, $a \neq 0$, $\Delta = 0$, and $b = 0$. It means that inside an inhomogeneous PCd phase, bubbles of the inhomogeneous phase with chiral spiral are allowed to exist and vice versa. By analogy, one might expect that in the more physically interesting case with $m_0 \neq 0$, the spatially inhomogeneous charged PCd phase would continue to be present at the phase diagram of the model (1), even if an arbitrary more general parameterization of condensates is used.

Appendix A: Roots of the equation $\eta^4 + A\eta^2 + B\eta + C = 0$

Using any program of analytical calculations, four roots of this equation can be presented in the following form:

$$\eta_1^\pm = \frac{1}{2}\sqrt{P} \pm \left(-2A - P - \frac{2B}{\sqrt{P}}\right)^{1/2}, \quad \eta_2^\pm = -\frac{1}{2}\sqrt{P} \pm \left(-2A - P + \frac{2B}{\sqrt{P}}\right)^{1/2}, \quad (\text{A1})$$

where

$$P = -\frac{2A}{3} + \frac{\sqrt[3]{2} R}{3Q} + \frac{Q}{3\sqrt[3]{2}}, \quad Q = \left(S + \sqrt{-4R^3 + S^2}\right)^{\frac{1}{3}}, \\ R = A^2 + 12C, \quad S = 2A^3 + 27B^2 - 72AC. \quad (\text{A2})$$

Appendix B: Derivation of formula (30)

Let us denote the integral in the left hand side of (30) by I (recall, there a is a real quantity).

It well-known that in quantum field theory any loop p_0 -integration is performed in the supposition that p_0 is a shorthand notation for $p_0 + i\varepsilon \cdot \text{sign}(p_0)$, where $\varepsilon \rightarrow 0_+$. In this case the causality of the theory is preserved. Taking this circumstance in mind, we see that in (30) the integration contour at $a > 0$ ($a < 0$) lies above (below) the singularity point $a - i\varepsilon \cdot \text{sign}(a)$ of an integrand function. Hence, it is possible to perform in (30) the Wick rotation of the integration contour and to direct it along the imaginary axis of the complex p_0 -plane. In thus obtained integral one can change an integration variable, $p_0 \rightarrow ip_0$. As a result, we come to the relation

$$I = i \int_{-\infty}^{\infty} dp_0 \ln(ip_0 - a) = i \int_0^{\infty} dp_0 \ln(ip_0 - a) + i \int_{-\infty}^0 dp_0 \ln(ip_0 - a). \quad (\text{B1})$$

In the last integral of (B1) one can again change an integration variable, $p_0 \rightarrow -p_0$. Hence,

$$I = i \int_0^{\infty} dp_0 \ln(ip_0 - a) + i \int_0^{\infty} dp_0 \ln(-ip_0 - a) = i \int_0^{\infty} dp_0 \ln(p_0^2 + a^2). \quad (\text{B2})$$

The last integral in (B2) can be easily taken using the integration by part method. Thus, up to an omitted infinite term which does not depend on a , we obtain $I = i\pi|a|$.

ACKNOWLEDGMENTS

The authors are grateful to Professor D. Ebert for the support of our investigations and for many fruitful discussions and to T.G. Khunjua for his interest to our problem.

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- [1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. D **112**, 345 (1961).
 - [2] M. Asakawa and K. Yazaki, Nucl. Phys. A **504**, 668 (1989); D. Ebert, H. Reinhardt and M.K. Volkov, Prog. Part. Nucl. Phys. **33**, 1 (1994).
 - [3] D. Ebert, K.G. Klimenko, M.A. Vdovichenko and A.S. Vshivtsev, Phys. Rev. D **61**, 025005 (2000); D. Ebert and K.G. Klimenko, Nucl. Phys. A **728**, 203 (2003).
 - [4] D.P. Menezes, M.B. Pinto, S.S. Avancini, A.P. Martinez and C. Providencia, Phys. Rev. C **79**, 035807 (2009); A. Ayala, A. Bashir, A. Raya and A. Sanchez, Phys. Rev. D **80**, 036005 (2009); N. Sadooghi, arXiv:0905.2097; E.J. Ferrer, V. de la Incera, J.P. Keith, I. Portillo and P.P. Springsteen, Phys. Rev. C **82**, 065802 (2010).
 - [5] A.J. Mizher, M.N. Chernodub and E.S. Fraga, Phys. Rev. D **82**, 105016 (2010); B. Chatterjee, H. Mishra and A. Mishra, Phys. Rev. D **84**, 014016 (2011).
 - [6] F. Preis, A. Rebhan and A. Schmitt, JHEP **1103**, 033 (2011); arXiv:1109.6904; M. D'Elia and F. Negro, Phys. Rev. D **83**, 114028 (2011); E.V. Gorbar, V.A. Miransky and I.A. Shovkovy, arXiv:1111.3401.
 - [7] M. Buballa, Phys. Rep. **407**, 205 (2005); I.A. Shovkovy, Found. Phys. **35**, 1309 (2005); M.G. Alford, A. Schmitt, K. Rajagopal, and T. Schäfer, Rev. Mod. Phys. **80**, 1455 (2008).
 - [8] D. Ebert, V.V. Khudiyakov, V.C. Zhukovsky and K.G. Klimenko, JETP Lett. **74**, 523 (2001); Phys. Rev. D **65**, 054024 (2002); D. Blaschke, D. Ebert, K.G. Klimenko, M.K. Volkov and V.L. Yudichev, Phys. Rev. D **70**, 014006 (2004); T. Fujihara, D. Kimura, T. Inagaki and A. Kvinikhidze, Phys. Rev. D **79**, 096008 (2009).
 - [9] E.J. Ferrer and V. de la Incera, Phys. Rev. D **76**, 045011 (2007); S. Fayazbakhsh and N. Sadooghi, Phys. Rev. D **82**, 045010 (2010); Phys. Rev. D **83**, 025026 (2011);

- [10] D.T. Son and M.A. Stephanov, Phys. Atom. Nucl. **64**, 834 (2001); M. Loewe and C. Villavicencio, Phys. Rev. D **67**, 074034 (2003); arXiv:1107.3859; L. He, M. Jin, and P. Zhuang, Phys. Rev. D **71**, 116001 (2005); D.C. Duarte, R.L.S. Farias and R.O. Ramos, Phys. Rev. D **84**, 083525 (2011); D. Ebert, K.G. Klimenko, A.V. Tyukov and V.C. Zhukovsky, Eur. Phys. J. C **58**, 57 (2008).
- [11] D. Ebert and K.G. Klimenko, J. Phys. G **32**, 599 (2006); Eur. Phys. J. C **46**, 771 (2006).
- [12] J.O. Andersen and T. Brauner, Phys. Rev. D **78**, 014030 (2008); J.O. Andersen and L. Kyllingstad, J. Phys. G **37**, 015003 (2009); Y. Jiang, K. Ren, T. Xia and P. Zhuang, arXiv:1104.0094.
- [13] C.f. Mu, L.y. He and Y.x. Liu, Phys. Rev. D **82**, 056006 (2010).
- [14] H. Abuki, R. Anglani, R. Gatto, G. Nardulli and M. Ruggieri, Phys. Rev. D **78**, 034034 (2008); H. Abuki, R. Anglani, R. Gatto, M. Pellicoro and M. Ruggieri, Phys. Rev. D **79**, 034032 (2009); R. Anglani, Acta Phys. Polon. Supp. **3**, 735 (2010).
- [15] D.J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
- [16] J. Feinberg, Annals Phys. **309**, 166 (2004); M. Thies, J. Phys. A **39**, 12707 (2006).
- [17] U. Wolff, Phys. Lett. B **157**, 303 (1985); T. Inagaki, T. Kouno, and T. Muta, Int. J. Mod. Phys. A **10**, 2241 (1995); S. Kanemura and H.-T. Sato, Mod. Phys. Lett. A **10**, 1777 (1995).
- [18] K.G. Klimenko, Theor. Math. Phys. **75**, 487 (1988).
- [19] A. Barducci, R. Casalbuoni, R. Gatto, M. Modugno, and G. Pettini, Phys. Rev. D **51**, 3042 (1995).
- [20] A. Chodos, H. Minakata, F. Cooper, A. Singh, and W. Mao, Phys. Rev. D **61**, 045011 (2000); K. Ohwa, Phys. Rev. D **65**, 085040 (2002).
- [21] A. Chodos and H. Minakata, Phys. Lett. A **191**, 39 (1994); H. Caldas, J.L. Kneur, M.B. Pinto and R.O. Ramos, Phys. Rev. B **77**, 205109 (2008); H. Caldas, Nucl. Phys. B **807**, 651 (2009); H. Caldas, arXiv:1106.0948.
- [22] V. Schon and M. Thies, arXiv:hep-th/0008175; Phys. Rev. D **62**, 096002 (2000); A. Brzoska and M. Thies, Phys. Rev. D **65**, 125001 (2002).
- [23] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
- [24] L.M. Abreu, A.P.C. Malbouisson and J.M.C. Malbouisson, Europhys. Lett. **90**, 11001 (2010); Phys. Rev. D **83**, 025001 (2011).
- [25] D. Ebert, K.G. Klimenko, A.V. Tyukov and V.C. Zhukovsky, Phys. Rev. D **78**, 045008 (2008).
- [26] D. Ebert and K.G. Klimenko, Phys. Rev. D **80**, 125013 (2009); V.C. Zhukovsky, K.G. Klimenko and T.G. Khunjua, Moscow Univ. Phys. Bull. **65**, 21 (2010).
- [27] D. Ebert and K.G. Klimenko, arXiv:0902.1861.
- [28] O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004); G. Basar, G.V. Dunne and M. Thies, Phys. Rev. D **79**, 105012 (2009); C. Boehmer and M. Thies, Phys. Rev. D **80**, 125038 (2009); J. Hofmann, Phys. Rev. D **82**, 125027 (2010).
- [29] F. Correa, G.V. Dunne and M.S. Plyushchay, Annals Phys. **324**, 2522 (2009); G. Basar and G.V. Dunne, JHEP **1101**, 127 (2011); M.S. Plyushchay, A. Arancibia and L.M. Nieto, Phys. Rev. D **83**, 065025 (2011); A. Arancibia and M.S. Plyushchay, arXiv:1111.0600; T. Kojo, arXiv:1106.2187.
- [30] D. Ebert, N.V. Gubina, K.G. Klimenko, S.G. Kurbanov and V.C. Zhukovsky, Phys. Rev. D **84**, 025004 (2011).
- [31] D.V. Deryagin, D.Y. Grigoriev and V.A. Rubakov, Int. J. Mod. Phys. A **7**, 659 (1992); M. Sadzikowski and W. Broniowski, Phys. Lett. B **488**, 63 (2000); W. Broniowski, arXiv:1110.4063.
- [32] E. Nakano and T. Tatsumi, Phys. Rev. D **71**, 114006 (2005).
- [33] D. Nickel, Phys. Rev. D **80**, 074025 (2009); S. Carignano, D. Nickel and M. Buballa, Phys. Rev. D **82**, 054009 (2010); H. Abuki, D. Ishibashi and K. Suzuki, arXiv:1109.1615.
- [34] S. Maedan, Prog. Theor. Phys. **123**, 285 (2010); A. Flachi, JHEP **1201**, 023 (2012).
- [35] I.E. Frolov, K.G. Klimenko and V.Ch. Zhukovsky, Phys. Rev. D **82**, 076002 (2010); Moscow Univ. Phys. Bull. **65**, 539 (2010); E.J. Ferrer, V. de la Incera and A. Sanchez, arXiv:1205.4492.
- [36] T. Kojo, Y. Hidaka, L. McLerran and R.D. Pisarski, Nucl. Phys. A **843**, 37 (2010).
- [37] G. Basar, G.V. Dunne and D.E. Kharzeev, Phys. Rev. Lett. **104**, 232301 (2010).
- [38] E.V. Gorbar, M. Hashimoto and V.A. Miransky, Phys. Rev. Lett. **96**, 022005 (2006); J.O. Andersen and T. Brauner, Phys. Rev. D **81**, 096004 (2010).
- [39] D. Ebert, T.G. Khunjua, K.G. Klimenko and V.C. Zhukovsky, arXiv:1106.2928 [hep-ph].